

II. Single Random Variables & Probability

Distributions.

Random variable:- A real variable x - whose value is determined by the outcome of a random experiment is called a random variable.

A random variable x can also be regarded as a real-value function defined on the sample space S of a random experiment such that for each point s of the sample space $f(s)$ is the probability of occurrence of the event represented by s .

Types of Random Variables: Random variables are

two types. They are,

① Discrete Random Variable.

② Continuous Random Variable.

* Discrete Random Variable:

A random variable x which can take only a finite numbers of discrete random variable. In other words, if the random variable takes the values only on the set $\{0, 1, 2, \dots, n\}$ is called a Discrete Random variable.

Ex:- The number of printing mistakes in each page of a book.

* Continuous Random variables:-

A random variable x which can take values continuously i.e. which takes all possible values in a given interval is called a continuous random variable.

Ex:- The height, age, weight of individuals.

Also temperature & time are continuous random variables.

* Probability Function of a Discrete Random variable:-

If for a discrete random variable x , the real valued function $p(x)$ is such that $p(x=x) = p(x)$ then $p(x)$ is called probability function (or) Probability density function of a discrete random variable x . Probability $p(x)$ gives the measure of probability for different values of x .

* Properties of a Probability function:-

If $p(x)$ is a probability function of a random variable x , Then it possesses the following properties are.

1. $p(x) \geq 0$ for all x .
2. $\sum p(x) = 1$. Summation is taken over for all values of x .
3. $p(x)$ cannot be negative for any value of x .

Probability Distribution Function:-

Let X be a random variable. Then the probability distribution function associated with X is a defined as the Probability that the outcome of an experiment will be one of the outcomes of X , which $X(s) \leq x$, $x \in \mathbb{R}$. That is the function $F(x)$, defined as $F_x(x) = P(X \leq x) = P\{s : X(s) \leq x\}$, $-\infty < x < \infty$ is called the distribution function of X .

Properties of Distribution function :-

1. If F is the distribution function of a random variable X and if $a < b$ then.

$$(i) P(a < X \leq b) = F(b) - F(a).$$

$$(ii) P(a \leq X \leq b) = P(X=a) + [F(b) - F(a)]$$

$$(iii) P(a < X < b) = [F(b) - F(a)] - P(X=b)$$

$$(iv) P(a \leq X < b) = [F(b) - F(a)] - P(X=b) + P(X=a)$$

2. All distribution functions are monotonically increasing, and lie b/w 0 and 1 that is if F is the distribution function of the random variable X , then

$$(i) 0 \leq F(x) \leq 1$$

$$(ii) F(x) \leq F(y) \text{ when } x < y.$$

$$\textcircled{3} (i) F(-\infty) = \lim_{x \rightarrow -\infty} F(x) = 0.$$

$$(ii) F(\infty) = \lim_{x \rightarrow \infty} F(x) = 1.$$

* Discrete Probability Distribution (Prob Mass Function):-

Discrete Probability distribution of a random variable

is the set of its possible values together with their

respective probabilities. Suppose x is a discrete

random variable with possible outcomes (values)

x_1, x_2, x_3, \dots . The probability of each possible

outcomes x_i is $P_i = P(x=x_i) = P(x_i)$ for $i=1, 2, 3, \dots$

If the numbers $P(x_i)$ $i=1, 2, 3, \dots$ satisfy the

two conditions.

(i) $P(x_i) > 0$ for all values of i ; $0 < P_i \leq 1$

(ii) $\sum P(x_i) = 1$, $i=1, 2, 3, \dots$

Then the function $P(x)$ is called the Probability

mass function of the random variable x and the

set $\{P(x_i)\}$, $i=1, 2, \dots$ is called the discrete Prob

distribution of the discrete random variable x .

Ex:- In tossing a coin two times.

$$S = \{\text{TT}, \text{HT}, \text{TH}, \text{HH}\}.$$

$P(x=0) = \text{Prob of getting two tails (no heads)}$

$$= P(\text{TT}) = \frac{1}{4}.$$

$P(x=1) = \text{Prob of getting one head}$

$$= P(\{\text{HT}, \text{TH}\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$P(x=2) = \text{Prob of getting two heads}$

$$= P(\text{HH}) = \frac{1}{4}.$$

Thus the total prob. 1 is distributed into three parts $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ according to whether $x=0$ or 1, or 2. This Prob. distribution is given in the following table.

$x=x_i$	0	1	2
$P(x=x_i)$	$1/4$	$1/2$	$1/4$

Cumulative Distribution Function of a Discrete Random variable?

There are many occasions in which it is of interest to know the probability that the value of a random variable is less than (or) equal to some real number x .

Suppose that x is a discrete random variable. Then the Discrete distribution Function (or) Cumulative Discrete Distribution Function $F(x)$ is defined by.

$$F(x) = P(x \leq x) = \sum_{(i, x_i \leq x)} P(x_i) = \sum_{i=1}^x P(x_i) \text{ where } x \text{ is any integer.}$$

Probability Density Function:

The probability density function $f_x(x)$ is defined as the derivative of the Probability distribution function $F_x(x)$ of the Random variable x .

$$\text{Thus } f_x(x) = \frac{d}{dx} [F_x(x)].$$

Expectation of a Probability distribution :-

Suppose a random variable 'x' assume the values x_1, x_2, \dots, x_n with respective Probability p_1, p_2, \dots, p_n . Then the expectation (or) expected values of 'x' denoted by $E(x)$ is defined as "Sum of Products of different values of x and the Corresponding Probabilities".

$$\text{i.e. } E(x) = p_1 x_1 + p_2 x_2 + \dots + p_n x_n.$$

$$E(x) = \sum_{i=1}^n p_i x_i.$$

$$\text{In general } E(g(x)) = \sum_{i=1}^n p_i g(x_i).$$

Some Important Results on Expectation:-

(i) If "x" is a random variable and 'k' is a constant, then $E(x+k) = E(x)+k$.

Proof :- Consider $E(x+k) = \sum_{i=1}^n p_i (x_i+k)$.

$$\begin{aligned} &= \sum_{i=1}^n p_i x_i + p_i k \\ &= \sum_{i=1}^n p_i x_i + k \sum_{i=1}^n p_i \\ &= E(x) + k(1). \quad \left[\because \sum_{i=1}^n p_i = 1 \right] \end{aligned}$$

$$E(x+k) = E(x) + k.$$

(ii) $E(kx) = kE(x)$

Proof :- Consider $E(kx) = \sum_{i=1}^n p_i (kx_i)$

$$= k \sum_{i=1}^n p_i x_i$$

$$E(kx) = k E(x).$$

(iii) $\epsilon(k) = k$.

Proof :- $\epsilon(k) = \sum_{i=1}^n k p_i = k \sum_{i=1}^n p_i = k(1) = k$

(iv) If x & y are two discrete random variable Then

$$\epsilon(x+y) = \epsilon(x) + \epsilon(y)$$

$$(v) \quad \epsilon(xy) = \epsilon(x) \cdot \epsilon(y)$$

$$\text{Note :- } \epsilon(x+y+z) = \epsilon(x) + \epsilon(y) + \epsilon(z)$$

$$\epsilon(ax+by) = [a\epsilon(x) + b\epsilon(y)]$$

Proof :- $\epsilon(x+y) = \sum_{i=1}^n p_i (x_i + y_i)$

$$= \sum_{i=1}^n p_i x_i + \sum_{i=1}^n p_i y_i$$

$$\epsilon(x+y) = \epsilon(x) + \epsilon(y).$$

(vi) $\epsilon(xy) = \epsilon(x) \cdot \epsilon(y)$

$$\epsilon(xy) = \sum_{i=1}^n p_i (x_i y_i)$$

$$= \sum_{i=1}^n p_i x_i \cdot \sum_{i=1}^n p_i y_i$$

$$= \epsilon(x) \cdot \epsilon(y).$$

$$[\because \sum_{i=1}^n p_i = 1]$$

For continuous random variable $f(x)$ $\epsilon(x) = \int x f(x) dx$

$\epsilon(xy) = \int xy f(x) dx$

$= \int x f(x) dx \cdot \int y f(x) dx$

Mean :- The mean value (μ) of a discrete distribution function is given by

$$\mu = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i} = \sum_{i=1}^n p_i x_i$$

unit - 2 Pg - 8/24

$$\mu = \sum_{i=1}^n p_i x_i = E(x)$$

Variance :- The variance (σ^2) of a discrete distribution function is given by

variance of the probability distribution of a random variable x is the mathematical expectation of $[x - E(x)]^2$. Then.

$$\text{var}(x) = E[x - E(x)]^2.$$

Another Form of variance :-

If x is a random variable, then the mathematical expectation of $(x - \mu)^2$ is defined to be the variance of the random variable x then,

$$\sigma^2 = E((x - \mu)^2). \Rightarrow \sum_{i=1}^n p_i (x_i - \mu)^2 = E(x_i^2 - 2\mu x_i + \mu^2) \\ = E(x^2 + \mu^2 - 2\mu x) = \sum x_i^2 p_i - 2\mu \sum x_i p_i + \mu^2 \sum p_i$$

$$= E(x^2) + E(\mu^2) - 2\mu E(x) = E(x^2) - 2\mu E(x) + \mu^2$$

$$= E(x^2) + E(\mu^2) - 2\mu(\mu) = E(x^2) - 2\mu(\mu) + \mu^2$$

$$= E(x^2) - E(\mu^2)$$

$$= E(x^2) - \mu^2$$

$$\text{var}(x) = E(x^2) - [E(x)]^2.$$

$$\text{var}(x) = E(x^2) - [E(x)]^2.$$

Standard Deviation :-

It is the positive square root of the variance.

$$\therefore S.D = (\text{---}) = \sqrt{\sum_{i=1}^n p_i x_i^2 - \mu^2} = \sqrt{E(x^2) - \mu^2} = E[(x - E(x))^2]$$

Some Important Results on Variance :-

1. Variance of Constant is zero $V(k) = 0$

2. If k is a constant, then $V(kx) = k^2 V(x)$

3. If x is a Random variable and k is a constant

Then $V(x+k) = V(x)$

4. If x is a discrete random variable then $V(ax+b) = a^2 V(x)$

5. If x & y are two independent random variable

then $V(x+y) = V(x) + V(y)$.

Problems :-

① Let x denote the number of heads in a single toss of 4 Fair Coins. Determine (i) $P(x < 2)$ (ii) $P(1 < x \leq 3)$.

Soln :- The required Probability distribution is.

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$4 \text{ heads} = \{HHHH\} = \frac{1}{16}$$

$$3 \text{ heads} = \{(HHHT), (HHTH), (HTHH), (THHH)\} = \frac{4}{16}$$

$$2 \text{ heads} = \{(HHTT), (HTHT), (HTHH), (THTH), (HTTH), (GHHH)\}$$

$$1 \text{ head} = \{(HTTT), (THTT), (TTHT), (TTTH)\} = \frac{4}{16} = \frac{6}{16}$$

$$0 \text{ heads} = \{(TTTT)\} = \frac{1}{16}$$

$$(i) P(X < 2) = P(X=0) + P(X=1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$(ii) P(1 < X \leq 3) = P(X=2) + P(X=3) = \frac{6}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}$$

(2) Two dice are thrown. Let X assign to each point (a, b) in S the maximum of its numbers i.e. $x(a, b) = \max(a, b)$. Find the probability distribution. X is a random variable with $x(s) = \{1, 2, 3, 4, 5, 6\}$. Also find the mean and variance of the distribution.

Soln :- The total no. of cases $= 6 \times 6 = 36$

The maximum number cases be 1, 2, 3, 4, 5, 6.

i.e. $x(s) = x(a, b) = \max(a, b)$.

The number 1 will appear only one case (1,1).

$$P(1) = P(X=1) = P(1,1) = \frac{1}{36}$$

$$P(2) = P(X=2) = P(1,2)(2,2)(2,1) = \frac{3}{36}$$

$$P(3) = P(X=3) = (1,3)(2,3)(3,3)(3,2)(3,1) = \frac{5}{36}$$

$$P(4) = P(X=4) = (1,4)(2,4)(3,4)(4,4)(4,3)(4,2)(4,1) = \frac{7}{36}$$

$$P(5) = P(X=5) = (1,5)(2,5)(3,5)(4,5)(5,5)(5,4)(5,3)(5,2)(5,1) = \frac{9}{36}$$

$$P(6) = P(X=6) = (1,6)(2,6)(3,6)(4,6)(5,6)(6,6)(6,5)(6,4)(6,3)(6,2) = \frac{11}{36}$$

\therefore The required discrete probability distribution is.

$x=x_i$	1	2	3	4	5	6
$P(x_i)$	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$

$$(i) \text{ Mean } \mu = \sum_{i=1}^6 P(x_i) x_i = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36}$$

$$= \frac{1}{36} (1 + 6 + 28 + 45 + 66) = \frac{161}{36} = 4.47$$

$$\begin{aligned}
 \text{(ii). Variance } (\sigma^2) &= \sum_{i=1}^6 p_i x_i^2 - \bar{x}^2 \\
 &= \frac{1}{36} (1)^2 + \frac{3}{36} (2)^2 + \frac{5}{36} (3)^2 + \frac{7}{36} (4)^2 + \frac{9}{36} (5)^2 + \frac{11}{36} (6)^2 - (4.47)^2 \\
 &= \frac{1}{36} (1+12+45+112+225+396) - (4.47)^2 \\
 &= \frac{791}{36} - 19.981 \\
 &= 21.97 - 19.981 = 1.9912.
 \end{aligned}$$

③ A random variable x has following Probability function

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Determine k

(ii) Evaluate $P(x \leq 6)$, $P(x \geq 6)$, $P(0 \leq x \leq 5)$ and $P(0 \leq x \leq 4)$

(iii) if $P(x \leq k) > \frac{1}{2}$, find the minimum value of k and

(iv) Determine the distribution function of x .

(v) Mean and (vi) variance.

Soln :-

(i) since $\sum_{x=0}^7 P(x) = 1$, we have.

$$k + 2k + 2k + 3k + k^2 + 7k^2 + 2k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$(10k-1)(k+1) = 0$$

$$k = \frac{1}{10} \quad \{k \neq -1 \text{ since } P(x) \geq 0\}$$

$$(ii) P(x \leq 6) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5)$$

$$= 0 + 2k + 2k + 3k + k^2$$

$$= 8k + k^2$$

$$= 8(0.1) + (0.1)^2$$

$$= 0.8 + 0.01$$

$$= 0.81$$

$$P(x \geq 6) = 1 - P(x \leq 6)$$

$$= 1 - 0.81$$

$$= 0.19$$

$$P(0 < x < 5) = P(x=1) + P(x=2) + P(x=3) + P(x=4)$$

$$= 1k + 2k + 2k + 3k = 8k$$

$$= 8(0.1) = 0.8$$

$$P(0 \leq x \leq 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$$

$$= 0 + k + 2k + 2k + 3k = 8k$$

$$= 8k = 8(0.1) = 0.8$$

(iii) The required minimum value of k is obtained

as below.

$$P(x \leq 1) = P(x=0) + P(x=1) -$$

$$= 0 + k = \frac{1}{10} = 0.1$$

$$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$$

$$= 0.01 + 2(0.1)$$

$$= 0.01 + 0.2 = 0.3$$

$$\text{(or)} = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} = 0.3$$

$$P(x \leq 3) = [P(x=0) + P(x=1) + P(x=2)] + P(x=3)$$

$$= 0.3 + \frac{2}{10} = 0.3 + 0.2 = 0.5$$

$$\begin{aligned}
 P(X \leq 4) &= P(X \leq 3) + P(X=4) \\
 &= 0.5 + \frac{3}{10} \\
 &= 0.5 + 0.3 = 0.8 > 0.5 = \frac{1}{2}.
 \end{aligned}$$

(iii) The minimum value of k for which $P(X \leq k) > \frac{1}{2}$ is

$$P(X \leq k) =$$

(iv) The distribution function of X is given by the

$$\text{function of } X \text{ is given by } F(x) = P(X \leq x)$$

$$\begin{aligned}
 0 & \quad 0 \\
 1 & \quad 1 - \frac{1}{10} = \frac{9}{10} \\
 2 & \quad 1 - \frac{3}{10} = \frac{7}{10} \\
 3 & \quad 1 - \frac{5}{10} = \frac{5}{10} \\
 4 & \quad 1 - \frac{8}{10} = \frac{2}{10} \\
 5 & \quad 1 - \frac{81}{100} = \frac{19}{100} \\
 6 & \quad 1 - \frac{83}{100} = \frac{17}{100} \\
 7 & \quad 1 - \frac{91}{100} = \frac{9}{100}
 \end{aligned}$$

$$(v) \text{ mean } \bar{\mu} = \sum_{i=0}^7 p_i x_i$$

$$\begin{aligned}
 &= 0(0) + 1(1k) + 2(2k) + 3(2k) + 4(3k) + 5(2k) + 6(2k) \\
 &\quad + 7(7k).
 \end{aligned}$$

$$= 66k^2 + 30k = \frac{66}{100} + \frac{30}{10} = 0.66 + 3 = 3.66.$$

$$(vi) \text{ Variance} = \sum_{i=0}^7 p_i x_i^2 - \bar{\mu}^2$$

$$\begin{aligned}
 &= k + 8k + 18k + 48k + 25k^2 + 72k^2 + 343k^2 + \\
 &\quad 49k - (3.66)^2 \\
 &= 440k^2 + 124k - (3.66)^2
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{440}{100} + \frac{124}{10} - (3.66)^2 \\
 &= 4.4 + 12.4 - 13.3956 \\
 &= 3.4044.
 \end{aligned}$$

④ From a lot of 10 items containing 3 defectives a sample of 4 items drawn at random. Let the random variable x denote the no. of defective items in the sample. Find the probability distribution of x when the sample is drawn without replacement.

Sol: x can take the values 0, 1, 2, (or) 3.

Given total no. of items = 10.

No. of good items = 7.

No. of defective items = 3.

$$P(x=0) = P(\text{no. defective}) = \frac{7C_4}{10C_4} = \frac{\frac{7!}{4!3!}}{\frac{10!}{6!}} = \frac{7!}{4!6!} = \frac{1}{6}.$$

$$= \frac{7!}{4!3!} \times \frac{4!6!}{10!} = \frac{1}{6}.$$

$P(x=1) = P(\text{one defective \& 3 good items})$

$$P(x=1) = \frac{3C_1 \times 7C_3}{10C_4} = \frac{3 \times 7!}{3!4!} \times \frac{4!6!}{10!} = \frac{1}{2}.$$

$P(x=2) = P(\text{2 defective and 2 good items})$

$$= \frac{3C_2 \times 7C_2}{10C_4} = \frac{3}{10}.$$

$P(x=3) = P(\text{3 defective \& 1 good item})$.

$$= \frac{3C_3 \times 7C_1}{10C_4} = \frac{7}{10C_4} = \frac{4!6!}{8 \times 9 \times 10} = \frac{1}{30}. \quad \text{Unit-2-Pg-14/24}$$

\therefore The probability distribution of random variable

X is as follows:-

x	0	1	2	3
$P(x)$	$1/6$	$1/2$	$3/10$	$1/30$

⑤ Let x denote the minimum of the two numbers that appears when a pair of fair dice is thrown once.

Determine the (i) Discrete Probability distribution.

(ii) Expectation (iii) Variance.

SOL :-

When two dice are thrown total no. of outcomes
 $= 36$.

The minimum number could be 1, 2, 3, 4, 5, 6.

For minimum 1, favourable cases are (1,1) (1,2) (2,1) (1,3)

(3,1) (1,4) (1,5) (1,6) (2,4) (2,5) (2,6) (3,4) (3,5) (3,6) (4,5) (4,6) (5,6).

$$\therefore P(x=1) = \frac{11}{36}.$$

$$P(x=2) = P(2,2) (2,3) (3,2) (2,4) (4,2) (2,5) (5,2) (2,6) (6,2) = \frac{9}{36}$$

$$P(x=3) = P(3,3) (3,4) (4,3) (3,5) (5,3) (3,6) (6,3) = \frac{7}{36}$$

$$P(x=4) = P(4,4) (4,5) (5,4) (4,6) (6,4) = \frac{5}{36}$$

$$P(x=5) = P(5,5) (5,6) (6,5) = \frac{3}{36}$$

$$P(x=6) = P(6,6) = \frac{1}{36}.$$

i). The probability distribution is.

x	1	2	3	4	5	6
$P(x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(ii) Expectation = $\mu = \text{mean} = \sum p_i x_i$

$$\begin{aligned} E(x) &= 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + 6 \cdot \frac{1}{36} \\ &= \frac{1}{36}(11+18+21+20+15+6) \\ &= \frac{91}{36} = 2.5278. \end{aligned}$$

(iii) Variance = $E(p_i x_i^2) - \mu^2$

$$\begin{aligned} E(p_i x_i^2) &= \frac{11}{36}(1)^2 + \frac{9}{36}(2)^2 + \frac{7}{36}(3)^2 + \frac{5}{36}(4)^2 + \frac{3}{36}(5)^2 + \frac{1}{36}(6)^2 \\ &\quad \text{(solution 3) add 300000} - \left(\frac{91}{36}\right)^2 \\ &= \frac{1}{36}(11+36+63+80+75+36) - \left(\frac{91}{36}\right)^2 \\ &= 8.3611 - 6.3898 = 1.9713. \end{aligned}$$

standard deviation, $\sigma = \sqrt{1.9713} = 1.404.$

⑥ Find the mean and variance of the uniform.

Probability distribution given by $f(x) = \frac{1}{n}$ for $x=1, 2, 3, \dots, n$

Solution: The Probability distribution is $P(x_i) = \frac{1}{n}$

$$f(x) = \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} = \frac{1}{n} (1+2+3+\dots+n)$$

(i) Mean = $\sum_{i=1}^n x_i f(x_i) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + \dots + n \cdot \frac{1}{n} = \frac{1}{n} \cdot n^2$

$$E(x) = \mu = \frac{1}{n} (1+2+\dots+n) = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

(ii) Variance = $\sum_{i=1}^n x_i^2 f(x_i) - \mu^2$

$$= 1^2 \cdot \frac{1}{n} + 2^2 \cdot \frac{1}{n} + 3^2 \cdot \frac{1}{n} + \dots + n^2 \cdot \frac{1}{n} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n} (1^2 + 2^2 + 3^2 + \dots + n^2) - \frac{1}{4} (n+1)^2$$

$$= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{4}(n+1)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \left(\frac{n+1}{2} \right) \left[\frac{2n+1}{3} - \frac{n+1}{2} \right]$$

$$= \frac{n+1}{12} (4n+2 - 3n-3) = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

Continuous Probability Distribution:-

When a random variable x takes every value in an interval, it gives rise to continuous distribution of x .

Ex:- age, height, weight, etc.

Probability Density Function:-

Consider the small interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$ of length dx round the point x . Let $f(x)$ be any continuous function of x so that $F(x)dx$ represent the probability that the variable x falls in the infinite

small interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$ symbolically it can be represented by:

$$P\left(x - \frac{dx}{2} \leq x \leq x + \frac{dx}{2}\right) = f(x)dx, \text{ Then } f(x) \text{ is}$$

called probability density function and the continuous curve $y=f(x)$ is known as probability density curve (or) simply probability curve.

As the probability for a variable value to lie in the interval ' dx ' is $f(x)dx$. So the probability for a variable value to fall in the finite interval

(a,b) is $\int_a^b f(x) dx = P(a < x < b)$ which represent the area between the curve $y=f(x)$.

Properties of the probability density Function:-

(i) $f(x) \geq 0 \quad \forall x \in R$.

(ii) $\int_{-\infty}^{\infty} f(x) dx = 1$.

(iii) The $P(c)$ is given by $P(c) = \int f(x) dx$

Cumulative Distribution Function (or) Prob'y distribution Function:-

The cumulative distribution (or) simply the

distribution function of a continuous random

variable 'x' is denoted by $F(x)$ and is defined as

$$F(x) = P(x \leq x) = \int_{-\infty}^x f(x) dx.$$

Thus, $F(x)$ gives the probability of that the value of $x \leq x$.

Properties:-

(i) $0 \leq F(x) \leq 1$.

Mean :- Mean of distribution is given by

$$\mu = E(x) = \int_{-\infty}^{\infty} f(x)x dx.$$

If 'x' is a random variable defined as (a,b) Then

$$\mu = E(x) = \int_a^b x f(x) dx.$$

In general $E(\phi(x)) = \int_{-\infty}^{\infty} \phi(x) f(x) dx$.

Median :- Median is the point which divides the entire distribution into two equal parts. Thus if $f(x)$ is defined on (a, b) (or) $[a, b]$ and ' M ' is the Median Then,

$$\int_a^M f(x) dx = \int_M^b f(x) dx = \frac{1}{2}.$$

$$\therefore \int_a^b f(x) dx = 1.$$

$$= \int_a^M f(x) dx + \int_M^b f(x) dx = 1$$

$$= \int_a^M f(x) dx = \int_M^b f(x) dx = \frac{1}{2}.$$

\therefore solving for ' M ' we get the median.

Mode :- Mode is the value of ' x ' for which $f(x)$ is maximum. Thus, mode is given by $f'(x)=0$ and $f''(x)<0$ for $a < x < b$.

Variance (σ^2) :-

Variance of a distribution is given by.

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx. \\ (\text{or})$$

$$= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

Mean deviation :-

Mean deviation about the mean (μ) is given by $\int_{-\infty}^{\infty} |x - \mu| f(x) dx.$

Q If a random variable has the Probability density function given as $f(x) = \begin{cases} 2e^{-2x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ find the probability it will take on a value.

(i) b/w 1 and 3. (ii) greater than 0.5.

SOL: (i) The Prob that a variate takes a value b/w 1 and 3 is given by.

$$P(1 \leq x \leq 3) = \int_1^3 f(x) dx = \int_1^3 2e^{-2x} dx.$$

$$= 2 \left[\frac{e^{-2x}}{-2} \right]_1^3$$

$$= - (e^{-6} - e^{-2}) = e^{-2} - e^{-6}.$$

(ii) The Prob that a variable takes a value greater than 0.5 is.

$$P(x \geq 0.5) = \int_{0.5}^{\infty} f(x) dx = \int_{0.5}^{\infty} 2e^{-2x} dx.$$

$$= 2 \left[\frac{e^{-2x}}{-2} \right]_{0.5}^{\infty}$$

$$= - (e^{-\infty} - e^{-1}) = - (0 - e^{-1}) = e^{-1}$$

② The Prob density $f(x)$ of a continuous random variable is given by $f(x) = C e^{-|x|}$ $-\infty < x < \infty$. Show that $C = \frac{1}{2}$ and find that the mean and variance of the distribution. Also find the probability that the variance lies b/w 0 and 4.

SOLN :- Given $f(x) = C e^{-|x|}$ $-\infty < x < \infty$.

we have $\int_{-\infty}^{\infty} f(x) dx = 1$

[since the total probability is Unity]

i.e. $\int_{-\infty}^{\infty} C e^{-|x|} dx = 1$

$$= 2C \int_0^{\infty} e^{-x} dx = 1 \quad [\because e^{-|x|} \text{ is an even function}]$$

$$= 2C \int_0^{\infty} e^{-x} dx = 1 \quad [\because \text{in } 0 \leq x \leq \infty, |x| = x]$$

$$= 2C (-e^{-x}) \Big|_0^{\infty} = 1.$$

$$= 2C (0 - 1) = 1$$

$$= 2C = 1$$

$$C = \frac{1}{2}.$$

Hence $f(x) = C e^{-|x|} = \frac{1}{2} e^{-|x|}$

(i) Mean of the distribution

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{2} e^{-|x|} dx = \frac{1}{2} \int_{-\infty}^{\infty} x e^{-|x|} dx.$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x e^{-|x|} dx = 0$$

since integrand is odd.

(ii) Variance of the distribution.

$$\text{Variance } \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

or variance $\sigma^2 = \int_{-\infty}^{\infty} (x - 0)^2 \frac{1}{2} e^{-|x|} dx$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|} dx$$

$$= 2 \cdot \frac{1}{2} \int_0^{\infty} x^2 e^{-|x|} dx$$

$$= \left[x^2 \frac{e^{-x}}{-1} - 2x \frac{e^{-x}}{-1} + 2 \frac{e^{-x}}{-1} \right]_0^{\infty}$$

$$= [0 - (0)] = 2 \cdot 1 = 2$$

(iii) The prob blw 0 and 4 = $P(0 \leq x \leq 4)$.

$$= \frac{1}{2} \int_0^4 e^{-|x|} dx = \frac{1}{2} \int_0^4 e^{-x} dx$$

$$= \frac{1}{2} \left[\frac{e^{-x}}{-1} \right]_0^4$$

$$= \frac{1}{2} [e^{-x}]_0^4$$

$$= \frac{1}{2} [e^{-4} - e^0]$$

$$= \frac{1}{2} [e^{-4} - 1] = \frac{1}{2}(1 - e^{-4})$$

$$= 0.4908 \text{ (nearly).}$$

③ A continuous random variable has the Prob density function $f(x) = \begin{cases} kxe^{-\lambda x} & \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{otherwise.} \end{cases}$

Determine (i) k (ii) Mean, (iii) Variance.

Sol :- (i) Since the total Prob is unity

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{i.e. } \int_{-\infty}^0 f(x) dx + \int_0^\infty f(x) dx = 1 - \int_{-\infty}^0 e^{-\lambda x} dx - \int_0^\infty e^{-\lambda x} dx$$

$$\begin{aligned} & \int_{-\infty}^0 0 \cdot dx + \int_0^\infty kx e^{-\lambda x} dx = 1 \quad \left[\frac{x e^{-\lambda x}}{-\lambda} - \int \frac{e^{-\lambda x}}{-\lambda} dx \right] \\ & k \int_0^\infty e^{-\lambda x} dx = 1 \quad \left[\frac{-1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda} \int e^{-\lambda x} dx \right] \\ & \therefore k \left[x \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 1 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) \right]_0^\infty = 1 \quad (0) \\ & = -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda} \frac{e^{-\lambda x}}{-\lambda} \\ & = k \left[\frac{-1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty \end{aligned}$$

$$k \left[(0-0) - (0 - \frac{1}{\lambda^2}) \right] = 1 \quad = k \left[(0-0) - (0 - \frac{1}{\lambda^2}) \right] = 1$$

$$\frac{k}{\lambda^2} = 1$$

$$\boxed{k = \lambda^2}$$

$$k = \lambda^2$$

$$\therefore \int u v = u \int v - \int u' \int v dx$$

Now $f(x)$ becomes.

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Mean of the distribution $\mu = \int_{-\infty}^{\infty} x f(x) dx$

$$\text{i.e. } \mu = \int_{-\infty}^0 f(x) dx + \int_0^\infty f(x) dx$$

$$= \int_{-\infty}^0 0 dx + \int_0^\infty \lambda^2 x \cdot x e^{-\lambda x} dx$$

$$= \lambda^2 \int_0^\infty x^2 e^{-\lambda x} dx$$

$$= \lambda^2 \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) \right]_0^\infty$$

$$= \lambda^2 \left[(0-0+0) - (0-0-\frac{2}{\lambda^3}) \right]$$

$$= \frac{2}{\lambda}$$

(iii) Variance of the distribution $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$

$$\sigma^2 = \int_0^\infty x^2 f(x) dx - \left(\frac{2}{\lambda} \right)^2 = \int_0^\infty \lambda^2 x \cdot x e^{-\lambda x} x^2 dx - \frac{4}{\lambda^2}$$

$$\begin{aligned}
 E(Y^2) &= \lambda^2 \int_0^\infty x^3 e^{-\lambda x} - \frac{4}{\lambda^2} \left(\lambda x \right) \left\{ \lambda x + \frac{6}{\lambda^2} \right\} e^{-\lambda x} dx \\
 &= \lambda^2 \int_0^\infty x^3 \left(e^{-\lambda x} \right) - 3x^2 \left(e^{-\lambda x} \right) + 6x \left(e^{-\lambda x} \right) + 6 \left(e^{-\lambda x} \right) dx - \frac{4}{\lambda^2} \\
 &= \lambda^2 \left[(0 - 0 + 0 - 0) - (0 - 0 + 0 - \frac{6}{\lambda^4}) \right] - \frac{4}{\lambda^2} \\
 &= \frac{6\lambda^2}{\lambda^4} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}
 \end{aligned}$$

- ④ If x is a continuous random variable and
 $y = ax + b$, prove that $E(y) = aE(x) + b$ and $V(y) = a^2V(x)$
 where V stands for variance and a, b are constants.

SOLN :- By definition

$$\begin{aligned}
 E(y) &= E(ax+b) \\
 &= \int_{-\infty}^{\infty} (ax+b) f(x) dx \\
 &= a \int_{-\infty}^{\infty} x f(x) dx + b \int_{-\infty}^{\infty} f(x) dx \\
 &= aE(x) + b
 \end{aligned}$$

$$\begin{aligned}
 E(y) &= aE(x) + b \quad \text{--- (1)} \\
 y &= ax + b \quad \text{--- (2)}
 \end{aligned}$$

$$\begin{aligned}
 (2) - (1) \quad y - E(y) &= ax + b - [aE(x) + b] \\
 &= a[x - E(x)] + b - b
 \end{aligned}$$

$$[y - E(y)]^2 = a^2 [x - E(x)]^2 \quad [\text{squaring on both sides}]$$

Taking expectation of both sides we get-

$$E\{[y - E(y)]^2\} = a^2 E\{[x - E(x)]^2\}$$

$$E[V(y)] = a^2 V(x)$$